ERF

The "erf" identity:

$$\int_0^c e^{-x^2} \cdot \partial x = \left(\frac{\pi^{1/2}}{2}\right) \cdot \text{erf}(c) \qquad (1)$$

The "erf" function itself is used to calculate the area under a bell-shaped (i.e. Gaussian) curve. Bell-shaped curves represent probability for certain phenomena such as the grade distribution of a class. The area of the curve can provide additional information; for example, the area over a certain range represents the probability of getting a certain letter grade. We also used the "erf" function in class to determine things like the average velocity of a gas molecule, or the probability that something is moving "up".

To do so we use this calculus identity:

$$\int_0^c x^2 \cdot e^{-a\cdot x^2} \cdot \partial x = \left(\frac{\pi}{16a^3}\right)^{1/2} \cdot \text{erf}\left(c \cdot a^{1/2}\right) - \left(\frac{c}{2a}\right) \cdot e^{-a\cdot c^2} \qquad (2)$$

We will use eq. (1) to derive eq. (2). However, eq. (1) doesn't have the extra x^2 component of eq. (2): e^{-x^2} vs. $\mathbf{x^2} \cdot e^{-a\cdot x^2}$. To deal with this, we use integration by parts:

$$\int_0^c f(x) \cdot \frac{\partial g(x)}{\partial x} \cdot \partial x = -\int_0^c \frac{\partial f(x)}{\partial x} \cdot g(x) \cdot \partial x + f(x) \cdot g(x)\Big]_0^c \qquad (3)$$

Applying eq. (3) to $\int_0^c x^2 \cdot e^{-a\cdot x^2} \cdot \partial x$ means that:

$$f(x) = x \quad \text{and} \quad \frac{\partial g(x)}{\partial x} = x \cdot e^{-a\cdot x^2}$$

which means $\frac{\partial f(x)}{\partial x} = 1$ and $g(x) = \left(\frac{-1}{2a}\right) \cdot e^{-a\cdot x^2}$. Plug these into eq. (3):

$$\int_0^c x^2 \cdot e^{-a\cdot x^2} \cdot \partial x = -\int_0^c \left(\frac{-1}{2a}\right) \cdot e^{-a\cdot x^2} \cdot \partial x + x \cdot \left(\frac{-1}{2a}\right) \cdot e^{-a\cdot x^2}\Big]_0^c$$

When we evaluate the 2nd term via the limits:

$$x \cdot \left(\frac{-1}{2a}\right) \cdot e^{-a\cdot x^2}\Big]_0^c = \left(\frac{-c}{2a}\right) \cdot e^{-a\cdot c^2} - 0 \cdot \left(\frac{-1}{2a}\right) \cdot e^{-0} = \left(\frac{-c}{2a}\right) \cdot e^{-a\cdot c^2}$$

Consequently:

$$\int_0^c x^2 \cdot e^{-a\cdot x^2} \cdot \partial x = \left(\frac{1}{2a}\right) \cdot \int_0^c e^{-a\cdot x^2} \cdot \partial x - \frac{c}{2a} \cdot e^{-a\cdot c^2} \qquad (4)$$

where $-\int_0^c \left(\frac{-1}{2a}\right) \cdot e^{-a\cdot x^2} \cdot \partial x$ was simplified as: $\left(\frac{1}{2a}\right) \cdot \int_0^c e^{-a\cdot x^2} \cdot \partial x$.

Almost done, except we have to solve: $\int_0^c e^{-a\cdot x^2}\cdot\partial x$, which is similar to eq. (1): $\int_0^c e^{-x^2}\cdot\partial x=\left(\frac{2}{\pi^{1/2}}\right)\cdot \mathrm{erf}(c)$, with the difference being that "a" in the argument of the exponential (i.e. e^{-x^2} vs. $e^{-\mathbf{a}\cdot x^2}$). We will handle that by substitution of variables to change the following:

$$\int_0^c e^{-a\cdot x^2}\cdot\partial x \qquad (5)$$

to look more like $\int_0^c e^{-x^2}\cdot\partial x$. First we substitute: $y^2=a\cdot x^2$, because this makes eq. (5) look more like (1): $\int_0^c e^{-a\cdot x^2}\cdot\partial x \rightarrow \int_0^c e^{-y^2}\cdot\partial x$. If you're having trouble seeing why this is a good move, note how the letter "a" has disappeared from the argument of the exponential. Next, we have to deal with the fact that the partial in $\int_0^c e^{-y^2}\cdot\partial x$ is ∂x, but what we need is ∂y. To resolve this, we use the Jacobian in eq. (6) to change ∂x into ∂y:

$$\int f(x)\cdot\partial x=\int f\big(g(y)\big)\cdot\left(\frac{\partial g(y)}{\partial y}\right)\cdot\partial y \qquad (6)$$

Why does this work? According to eq. (6), $g(y)=x$. Consequently:

$$\left(\frac{\partial g(y)}{\partial y}\right)\cdot\partial y=\left(\frac{\partial x}{\partial y}\right)\cdot\partial y=\partial x\cdot\left(\frac{\partial y}{\partial y}\right)=\partial x$$

where we used $\frac{\partial y}{\partial y}=1$. To summarize:

$$\int f\big(g(y)\big)\cdot\left(\frac{\partial g(y)}{\partial y}\right)\cdot\partial y=\int f(x)\cdot\left(\frac{\partial x}{\partial y}\right)\cdot\partial y=\int f(x)\cdot\partial x\cdot\left(\frac{\partial y}{\partial y}\right)=\int f(x)\cdot\partial x$$

The hard part of using the substitution of variables eq. (6) is to identify $g(y)$. As we already decided that $y=a\cdot x^2$, and since $x=g(y)$, we just have to turn $y^2=a\cdot x^2$ into a function of x. First divide out the "a": $\frac{y^2}{a}=x^2$, and next, take the square root of both sides: $\frac{y}{a^{1/2}}=x=g(y)$ which allows us to evaluate $\frac{\partial g(y)}{\partial y}$:

$$\frac{\partial g(y)}{\partial y}=\frac{\partial\left(y/a^{1/2}\right)}{\partial y}=\frac{1}{a^{1/2}}$$

Putting this all together:

$$\int f(x)\cdot\partial x=\int e^{-a\cdot x^2}\cdot\partial x=\int e^{-a\cdot\left(\frac{y^2}{a}\right)}\cdot\left(\frac{1}{a^{1/2}}\right)\cdot\partial y=\left(\frac{1}{a^{1/2}}\right)\cdot\int e^{-y^2}\cdot\partial y$$

Last bit, we have to deal with the limits of integration:

$$\int_0^c e^{-a\cdot x^2}\cdot\partial x \rightarrow \left(\frac{1}{a^{1/2}}\right)\cdot\int_?^? e^{-y^2}\cdot\partial y$$

The limits on the left side are for “x”, but we have to make them for “y” on the right side. To do so, note the lower limit x=0, when plugged into: $\frac{y}{a^{1/2}} = x$, makes $y = 0$. The upper limit of x=c, when plugged into: $\frac{y}{a^{1/2}} = x$, is $y = c\cdot a^{1/2}$. Thus:

$$\int_0^c e^{-a\cdot x^2}\cdot\partial x = \left(\frac{1}{a^{1/2}}\right)\cdot\int_0^{c\cdot a^{1/2}} e^{-y^2}\cdot\partial y$$

At this point we can use the identity eq. (1): $\int_0^c e^{-x^2}\cdot\partial x = \left(\frac{\pi^{1/2}}{2}\right)\cdot\mathrm{erf}(c)$ and apply it to: $\left(\frac{1}{a^{1/2}}\right)\cdot\int_0^{a^{1/2}\cdot c} e^{-y^2}\cdot\partial y$ to see that the answer is:

$$\int_0^c e^{-a\cdot x^2}\cdot\partial x = \left(\frac{1}{a^{1/2}}\right)\cdot\int_0^{c\cdot a^{1/2}} e^{-y^2}\cdot\partial y = \left(\frac{1}{a^{1/2}}\frac{\pi^{1/2}}{2}\right)\cdot\mathrm{erf}\left(c\cdot a^{1/2}\right) = \left(\frac{\pi^{1/2}}{2a^{1/2}}\right)\cdot\mathrm{erf}\left(c\cdot a^{1/2}\right) \quad (7)$$

Finally, we see that, starting from eq. (4):

$$\int_0^c x^2\cdot e^{-a\cdot x^2}\cdot\partial x = \left(\frac{1}{2a}\right)\cdot\int_0^c e^{-a\cdot x^2}\,\partial x - \frac{c}{2a}\cdot e^{-a\cdot c^2}$$

Substitute in eq. (7): $\int_0^c e^{-a\cdot x^2}\cdot\partial x = \left(\frac{\pi^{1/2}}{2a^{1/2}}\right)\cdot\mathrm{erf}\left(c\cdot a^{1/2}\right)$ makes (4) become:

$$\int_0^c x^2\cdot e^{-a\cdot x^2}\cdot\partial x = \left(\frac{1}{2a}\right)\cdot\left(\frac{\pi^{1/2}}{2a^{1/2}}\right)\cdot\mathrm{erf}\left(c\cdot a^{1/2}\right) - \frac{c}{2a}\cdot e^{-a\cdot c^2}$$

When you simplify it: $\int_0^c x^2\cdot e^{-a\cdot x^2}\cdot\partial x = \left(\frac{\pi}{16a^3}\right)^{1/2}\cdot\mathrm{erf}\left(c\cdot a^{1/2}\right) - \frac{c}{2a}\cdot e^{-a\cdot c^2}$

This is equation (2), which is what we set out to prove.